arXiv:math/0402095v1 [math.NT] 6 Feb 2004

POSITIVITY OF HEIGHTS OF SEMI-STABLE VARIETIES

ROBERTO G. FERRETTI

1. Introduction

Let K be a number field and $\mathcal{O}_K$ its ring of integers. Let $\mathcal{E}$ be an $\mathcal{O}_K$-module of rank $N+1$ in $\mathbb{P}(\mathcal{E}^\vee)$, the projective space representing lines in $\mathcal{E}^\vee$. For all closed subvarieties $X \subseteq \mathbb{P}(\mathcal{E}_K^\vee)$ of dimension d let $\deg(X)$ be its degree with respect to the canonical line bundle $\mathcal{O}(1)$ of $\mathbb{P}(\mathcal{E}_K^\vee)$. If $\mathcal{E}$ is endowed with the structure of hermitian vector bundle over $\operatorname{Spec}(\mathcal{O}_K)$ we can define the Arakelov degree $\widehat{\deg}\left(\overline{\mathcal{E}}\right)$ and the Faltings height $h_{\mathcal{E}}(X)$ (see **3.1**).

Let $\mathcal{V}$ be the direct limit of the set $\mathcal{V}_K$ of $\mathcal{O}_K$-modules $\mathcal{E}$ with an identity $\mathcal{E}_K \cong K^{N+1}$ as K varies in the set of number fields. Let $X \subseteq \mathbb{P}^N$ be a closed irreducible projective variety of dimension d defined over $\overline{\mathbb{Q}}$. We define

$$\hat{h}(X) = \inf_{\mathcal{E}\in\mathcal{V}} \left(\frac{h_{\mathcal{E}}(X)}{(d+1)\deg(X)} - \frac{\widehat{\deg}(\overline{\mathcal{E}})}{N+1} \right).$$

According to [1, Théorème 1] if the Chow point of X is *semistable* (see **2.4**) then there exists a non-negative constant C such that

$$\hat{h}(X) \geqslant -C. \tag{1.1}$$

This can be shown using the fact that for $\mathcal{E} \in \mathcal{V}$ the expression $h_{\mathcal{E}}(X)/(d+1)\deg(X)$ is bounded below in terms of the height of a system of generators of the rings of invariants $(\operatorname{Sym} E)^{SL(E)}$ (see also [12]). Moreover, one can even show that $h_{\mathcal{E}}(X)/(d+1)\deg(X)$ is bounded below by the average of the successive minima of $\mathcal{E}$ (see [22]). These results give a lower bounds of $\hat{h}(X)$ that depends on the field K of definition of $\mathcal{E}$ and X. The independence of the constant C from the field of definition has been shown by Zhang in [26], using his absolute successive minima [27, 5.2], and independently by Bost in [2], using the ring of invariants.

We will prove in this section a conjecture of Zhang [26, 4.3] that the height $\hat{h}(X)$ of a semistable variety X is non-negative, as it is true in the function field case (see [5], [1]). The idea of the proof is the same of that for the main theorem of [22], the main difference

Date: February 1, 2008.

1991 *Mathematics Subject Classification.* 11G50,14G40,14L24.

Key words and phrases. Arakelov Geometry, Geometric Invariant Theory.

relies in the fact that here we make direct use of Zhang's absolute successive minima of X (defined in [25]) at the place of Zhang's minima of the projective space.

Theorem 1.1. *Let us suppose that X is semistable. Then*

$$\hat{h}(X) \geqslant 0. \tag{1.2}$$

In order to prove this inequality we have to introduce the *degree of contact*. This is a birational invariant first considered in the context of Geometric Invariant Theory (see [19]), which has recently found important applications in the domain of diophantine approximations. We refer to [19], and [8, §3] for exhaustive and detailed discussions of the main properties the degree of contact (see also [9], [10] and the references therein).
As a byproduct, we are able to show, for all hermitian $\mathcal{O}_K$-module $\mathcal{E}$, new lower bounds for the normalized Faltings height $h_{\mathcal{E}}(X)$, (see (3.1)). For instance, when X is a generic $K3$ surface, i.e. a $K3$ surface whose Picard group has rank one, then X is semistable (see Corollary 2.10 and [18]), so Theorem 1.1 holds true. This implies that the height $h_{\mathcal{E}}(X)$ is non-negative for all hermitian $\mathcal{O}_K$-module $\mathcal{E}$. However, Corollary 3.5 provides a lower bound for the Faltings height in terms of a linear combination of Zhang's absolute minima which is stronger than that given by the semistability of X.
The seminal work of Zhang [26] has even contributed to lay down the basis for the proof by Phong and Sturm [21] of a conjecture formulated by Donaldson in his work [7] on Yau's conjecture. However, Donaldson, Phong and Sturm were concerned about one direction of Yau' conjecture, namely on the proof of the semistability of X under the assumption of the existence of a Kähler-Einstein metric. It would be interesting to see if the methodology developed in this note could be useful to better understand the other direction of Yau's conjecture.

2. Degree of contact

2.1. Let E be a finite dimensional vector space defined over a number field K. A *weight function* is a map $w : E \to \mathbb{R} \cup \{-\infty\}$ satisfying:

(1) $w(x) = -\infty$, if and only if $x = 0$,
(2) for all $t \in K^*$ and all $x \in E$, $w(t \cdot x) = w(x)$,
(3) for all $x, y \in E$, $w(x + y) \leqslant \max\{w(x), w(y)\}$.

For all non-negative real numbers α the set $F^\alpha = \{x \in E \ : \ w(x) \leqslant \alpha\}$ is a subspace of E, and $F^\alpha \subseteq F^\beta$ whenever $\alpha \leqslant \beta$. Varying α, we get then an exhaustive ($\cup_{\alpha\in\mathbb{R}} F^\alpha = E$) and separated ($\cap_{\alpha\in\mathbb{R}} F^\alpha = \{0\}$) filtration $\mathcal{F}$ of subspaces of E. We say that a basis $l_0, \cdots, l_N$ of E is *adapted to the filtration* $\mathcal{F}$, if for all $\alpha \in \mathbb{R}$

$$F^\alpha = \bigoplus_{w(l_j)\leqslant\alpha} K \cdot l_j.$$

We number the elements of this basis so that $w(l_0) \geqslant w(l_1) \geqslant \cdots \geqslant w(l_N)$. For $h = 0, \cdots, N$ define $r_h := w(l_h)$ and $\mathbf{r} = (r_0, \cdots, r_N)$. The vector $\mathbf{r}$ is obviously independent of the chosen basis. This construction gives a bijective map between the set of weight functions on E and the set of couples $(\mathcal{F}, \mathbf{r})$, with $\mathcal{F}$ a filtration of subspaces of E, and $\mathbf{r} = (r_0, \cdots, r_N) \in \mathbb{R}^{N+1}_{\geqslant 0}$ with $r_0 \geqslant \cdots \geqslant r_N$. Indeed, to such a pair $(\mathcal{F}, \mathbf{r})$ we associate a weight function w on E as follows: Put $w(0) = -\infty$ and for $x \in E \setminus \{0\}$ define $w(x)$ as the smallest r_i such that $x \in F^{r_i}$. Equivalently, given a basis $l_0, \cdots, l_N$ of E adapted to the filtration $\mathcal{F}$, write $x = x_0 l_0 + \cdots x_N l_N$. Then $w(x)$ is the greatest r_i for which $x_i \neq 0$. Notice that the value $w(x)$ does not change much if we dilate $\mathbf{r}$ or perturb it a little bit. We can always find an integer valued weight function $\tilde{w}$ supported on the same filtration $\mathcal{F}$ of w, such that for some sufficiently small $\varepsilon > 0$ and some positive integer m we have $mw \leqslant \tilde{w} \leqslant m(1+\varepsilon)w$. This enables us to reduce most of the computations to weight functions with integer values.

2.2. Weight functions satisfy several functorial relations. Let w be a weight function on E with non-negative integer weights $\mathbf{r} \in \mathbb{Z}^{N+1}$ and associated filtration $\mathcal{F}$. Consider a subspace $F \subseteq E$. The restriction $w|_F$ of w on F defines a weight function on F. Further, w induces a weight function on the quotient E/F, mapping l to the minimum of the weights of the elements $x \in E$ with $\pi(x) = l$, where $\pi : E \to E/F$ is the canonical projection. An element $h \in E^\vee$ is a linear functional $h : E \to K$. If $h \neq 0$ we define the weight of h as minus the weight of the line $E/\ker(h)$.

Given two vector spaces E_1, and E_2 over K, endowed with weight functions w_1, w_2 respectively, we define a weight function w on $E_1 \oplus E_2$ by

$$e_1 \oplus e_2 \mapsto \max\{w_1(e_1), w_2(e_2)\}. \tag{2.1}$$

Moreover, on the tensor product $E_1 \otimes E_2$ we define a weight function w by $w(e_1 \otimes e_2) = w_1(e_1) + w_2(e_2)$. Let m be a positive integer. The symmetric group of order m operates on the m-th tensor power $E^{\otimes m}$ by permuting the factors. The m-th symmetric power of E, denoted by $\mathrm{Sym}^m E$, is the maximal subspace of $E^{\otimes m}$ invariant under this operation. Whence by restriction we can canonically define a weight function on $\mathrm{Sym}^m E$. The exterior power $\bigwedge^m E$ is the quotient of $E^{\otimes m}$ by $\mathrm{Sym}^m E$, therefore on this space there exists a canonical induced weight function.

2.3. Let w be a weight function on E with integer weights $\mathbf{r} \in \mathbb{Z}^{N+1}$, and let $\mathcal{F}$ be the associated filtration. Let X be a projective absolutely irreducible scheme of dimension d and defined over K embedded into the projective space $\mathbb{P}(E^\vee)$ representing lines of the dual vector space $E^\vee$. If m is large enough, say $m \geqslant m_0$, the cup product map

$$\varphi_m : \mathrm{Sym}^m(E) \to H^0(X, \mathcal{O}(m)) \tag{2.2}$$

is surjective. Therefore $H^0(X,\mathcal{O}(m))$ can be identified with a quotient of $\mathrm{Sym}^m(E)$. As in **2.2** w induces then a weight function on $H^0(X,\mathcal{O}(m))$, and on the one-dimensional space $\bigwedge^{h^0(X,\mathcal{O}(m))} H^0(X,\mathcal{O}(m))$. We denote by $w(X,m)$ the weight of this line, which is well defined by the homogeneity property (2) of weight functions. There exists an integer $e_w(X)$ such that when m goes to infinity

$$w(X,m) \;=\; e_w(X)\frac{m^{d+1}}{(d+1)!}+O(m^d), \tag{2.3}$$

(see [19, Proposition 2.11], or [8, $\S3$]). The number $e_w(X)$ is called *degree of contact* of X with respect to the weighted filtration associated to the weight function w. We extend this definition by linearity to cycles, and by approximating to real weights $\mathbf{r}\in\mathbb{R}^{N+1}$.
In the last years appeared several articles in diophantine approximations that make a wide use of the degree of contact (see [8],[9], and [10]). In these articles the main properties of the degree of contact are discussed in detail. We refer to them for a further thorough analysis of the degree of contact (see also [19], [17], [16]).

2.4. Each suitably generic element $\mathbf{h}=(h_0,\cdots,h_d)\in\mathbb{P}(E)^{d+1}$ defines naturally a $(N-d-1)$-dimensional linear subspace $L_{\mathbf{h}}\subseteq\mathbb{P}(E^\vee)$. Consider the set $Z(X)$ of all $(d+1)$-tuples $\mathbf{h}\in\mathbb{P}(E)^{d+1}$, such that $L_{\mathbf{h}}(\overline{\mathbb{Q}})\cap X(\overline{\mathbb{Q}})\neq\emptyset$, where $L_{\mathbf{h}}$ has dimension $N-d-1$. Then $Z(X)$ is an irreducible hypersurface of multidegree $(\deg(X),\cdots,\deg(X))$ (see [15, Thm. IV, p. 41]). The hypersurface $Z(X)$ turns out to be given by an up to a constant unique polynomial element $F_X\in V$, where

$$V \;=\; \left[\left(\mathrm{Sym}^{\deg(X)}E\right)^{\otimes(d+1)}\right]^\vee . \tag{2.4}$$

This is a so called *(Cayley-Bertini-van der Waerden-) Chow form* of X. By definition it has that property that $F_X(h_0,\cdots,h_d)=0$ if and only if X and the hyperplanes given by the vanishing of the linear forms h_i $(i=0,\cdots,d)$ have a point in common over $\overline{\mathbb{Q}}$.

2.5. As in **2.2** w induces a weight function on V, again denoted by w. From [19, Proposition 2.11] (see also [8, Theorem 4.1]) we know that the degree of contact corresponds to minus the weight of the Chow point:

$$e_w(X)=-w(F_X). \tag{2.5}$$

Indeed, using the terminology of [19], the *"n.l.c. of r_n^V,"* (the degree of contact) corresponds to the *"λ-weight a_V of ϕ_V"* (minus the weight of the Chow form in our notation).
We say that the variety X is *Chow-semistable* (or simply semistable) if for all weight functions w on E we have

$$\frac{e_w(X)}{(d+1)\deg(X)}\leqslant\frac{1}{N+1}\sum_{i=0}^{N} r_i.$$

According to the Hilbert-Mumford criterion (see [19]), this is equivalent to say that the Zariski closure of the orbit of a representative of F_X in V under $SL(E)$ does not contain 0.

2.6. Let $l_0, \cdots, l_N$ be a basis of E adapted to the filtration associated to the weight function w, which identifies then E with K^{N+1}, and define $T = \binom{N+1}{d+1}$. Given the blocks of variables $h_p = (h_{p0}, \cdots, h_{pN})$ $(p = 0, \cdots, d)$ we define for each subset $I_k = \{i_{k0}, \cdots, i_{kd}\}$ of $\{0, \cdots, N\}$ with $i_{k0} < \cdots < i_{kd}$ the *bracket* $[I_k] = [i_{k0} \cdots i_{kd}] = \det(h_{p,i_{kq}})_{p,q=0,\cdots,d}$, for $k = 1, \cdots, T$. From [15, Thm. IV, p.41] it follows that the Chow form F_X can be expressed as a polynomial in terms of such brackets. We expand F_X as a sum of monomials of brackets

$$F_X = \sum_{\mathbf{j}=(j_1,\cdots,j_T)\in\mathcal{J}} a_{\mathbf{j}} [I_1]^{j_1} \cdots [I_T]^{j_T}$$

where $a_{\mathbf{j}} \neq 0$, and $|\mathbf{j}| = \deg(X)$ for $\mathbf{j} \in \mathcal{J}$. Then if w is a weight function given by the weights $r_0 \geqslant \cdots \geqslant r_N \geqslant 0$ we have

$$e_w(X) = \min_{\mathbf{j}\in\mathcal{J}} \sum_{i=1}^{T} j_i \left(\sum_{k\in I_i} r_k \right).$$

2.7. Suppose that the hyperplanes defined by the vanishing of the linear forms $l_{N-d}, \cdots, l_N$ do not have a common point on X defined over $\overline{\mathbb{Q}}$. If X is linear, this means precisely that the restrictions to X of the linear forms $l_{N-d}, \cdots, l_N$ are linearly independent. We notice that for given real numbers $0 \leqslant c_0 \leqslant \cdots \leqslant c_N$ and for $\mathbf{j} \in \mathcal{J}$ we have

$$\sum_{i=1}^{T} j_i \left(\sum_{k\in I_i} c_k \right) \leqslant \deg(X)\Big(c_{N-d} + \cdots + c_N\Big). \tag{2.6}$$

Further, due to our assumption that X and the zero set of $l_{N-d}, \cdots, l_N$ do not meet, we have $F_X(l_{N-d}, \cdots, l_N) \neq 0$. Therefore, F_X must contain the monomial $[N-d \cdots N]^{\deg(X)}$. This implies that among the terms $\sum_{i=1}^{T} j_i \left(\sum_{k\in I_i} c_k\right)$, with $\mathbf{j} \in \mathcal{J}$ we have $\deg(X)(c_{N-d} + \cdots + c_N)$. But this is the largest among all the terms (2.6), whence

$$\max_{\mathbf{j}\in\mathcal{J}} \sum_{i=1}^{T} j_i \left(\sum_{k\in I_i} c_k \right) = \deg(X)\Big(c_{N-d} + \cdots + c_N\Big).$$

Remember that w is a weight function given by the basis $l_0, \cdots, l_N$ and weights $r_0 \geqslant \cdots \geqslant r_N \geqslant 0$. We define $c_i = r_0 - r_i$, for $i = 0, \cdots, N$. We have

$$\begin{aligned} e_w(X) &= \min_{\mathbf{j}\in\mathcal{J}} \sum_{i=1}^{T} j_i \left(\sum_{k\in I_i} r_k \right) \\ &= \deg(X)(d+1)r_0 - \max_{\mathbf{j}\in\mathcal{J}} \sum_{i=1}^{T} j_i \left(\sum_{k\in I_i} c_k \right) \\ &= \deg(X)(d+1)r_0 - \deg(X)\Big(r_0 - r_{N-d} + \cdots + r_0 - r_N\Big) \\ &= \deg(X)\Big(r_{N-d} + \cdots + r_N\Big). \qquad (2.7) \end{aligned}$$

2.8. Let X be an absolute irreducible projective surface and C a *pseudo ample* divisor on X, i.e. a divisor such that the linear series $|C|$ has no fixed components and the associated map $\phi_C : X \to \mathbb{P}(H^0(X, mC)^\vee)$ is birational onto its image, for m sufficiently large. For $E = H^0(X, mC)$ with $\dim(E) = N+1$ let w be a weight function on E whith associated filtration

$$\mathcal{F}: \quad E = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_N \supseteq \{0\}. \qquad (2.8)$$

Suppose that there exists a blow up $\pi : B \to X$ on which C has a proper transform $\tilde{C}$, and such that for each $i = 0, \cdots, N$ the pullbacks of the sections in V_i generate an invertible sub-sheaf $\mathcal{O}_B(C_i)$ of $\mathcal{O}(\tilde{C})$. According to [11, 4.4] $\mathcal{O}_B(C_i) = \pi^*\mathcal{O}(C) \otimes \pi^{-1}(J_i)$, where J_i is the ideal sheaf defining the base locus of $|V_i|$. This means that the number C_i^2 is the degree of the projection of X onto $\mathbb{P}(V_i^\vee)$. Define

$$e_{ij} = C^2 - C_i \cdot C_j, \quad e_j = C^2 - C_j^2.$$

These numbers are independent of the choice of B. Note that for all $i = 0, \cdots, N$ the number e_i measures the drop in degree on projection to $\mathbb{P}(V_i^\vee)$. Let us identify X with the image of the birational map ϕ_C. Let $r_0 \geqslant \cdots \geqslant r_N \geqslant 0$ be the weights associated to the weight function w. Then for all sequences of integers $J = (j_0, \cdots, j_l)$ with $0 = j_0 < j_1 < \cdots < j_l = N$ [10, Proposition 2.10] yields

$$e_w(X) \leqslant \sum_{k=0}^{l-1} (r_{j_k} - r_{j_{k+1}}) \Big(e_{j_k} + e_{j_k, j_{k+1}} + e_{j_{k+1}} \Big) =: S_J. \qquad (2.9)$$

Assume now that X is a $K3$ surface over K, C a pseudo ample divisor on X, and $m = 1$. Then

$$h^0(X, C) = \frac{C^2}{2} + 2 \quad \text{and} \quad h^1(X, C) = 0. \qquad (2.10)$$

Further C has no base points and $\phi_C(X)$ is projectively normal ([23, 2.6], [23, 3.2], [23, 6.1]). If we assume that no curve is contained in the base locus of V_i, for all $i = 0, \cdots, N$, and that $r_N = 0$, then from [18, Lemma 5] and [10, Proposition 2.11] we have

$$e_w(X) \;\leqslant\; -4r_0 + 6\sum_{i=0}^{N} r_i. \tag{2.11}$$

We are moreover able to prove the following tightening of (2.11).

Proposition 2.9. *Assume now that X is a $K3$ surface over K, C a pseudo ample divisor on X, and that no curve is contained in the base locus of V_i, for all $i = 0, \cdots, N$. Then if $r_N = 0$ we have*

$$e_w(X) \leqslant \min\left\{-4r_0 + 6\sum_{i=0}^{N} r_i, 2(N-1)r_0\right\}.$$

Proof. Remember that for $i = 0, \cdots, N$, the number e_i is the amount by which the degree of X in $\mathbb{P}(E^\vee)$ is greater that that of its image under the projection onto $\mathbb{P}(V_i^\vee)$. In any case

$$e_i \;\leqslant\; C^2.$$

By Riemann-Roch (2.10) we have $N+1 = h^0(X, C) = \frac{C^2}{2} + 2$, which implies

$$e_i \;\leqslant\; 2(N-1). \tag{2.12}$$

Let us now consider the inequality (2.9) with $J = (0, N)$. Then from (2.12) we get

$$e_w(X) \leqslant S_J = (r_0 - r_N)\Big(e_0 + e_{0N} + e_N\Big) \leqslant 2(N-1)r_0$$

Together with [10, Proposition 2.11] this concludes the proof. □

We recover here the main result of [18].

Corollary 2.10. *Let X be a $K3$ surface whose Picard group has rank 1 and C be a primitive divisor class on X. Then X is semistable.*

Proof. Let w be any weight function on $E = H^0(X, C)$ and $r_0 \geqslant \cdots \geqslant r_N \geqslant 0$ be the associated weights. We can assume without restriction that $r_N = 0$. Let us first suppose that

$$r_0 \geqslant \frac{3}{N+1}\sum_{i=0}^{N} r_i.$$

Together with Riemann-Roch's formula (2.10) and Proposition 2.9 this implies

$$\frac{e_w(X)}{(\dim X + 1)\deg(X)} \;\leqslant\; \frac{-4r_0 + 6\sum_{i=0}^{N} r_i}{6(N-1)} \leqslant \frac{1}{N+1}\sum_{i=0}^{N} r_i.$$

We assume now

$$r_0 \leqslant \frac{3}{N+1} \sum_{i=0}^{N} r_i.$$

Then again Riemann-Roch's formula (2.10) and Proposition 2.9 imply

$$\frac{e_w(X)}{(\dim X + 1)\deg(X)} \leqslant \frac{2(N-1)r_0}{6(N-1)} \leqslant \frac{1}{N+1} \sum_{i=0}^{N} r_i,$$

which concludes the proof. □

Remark 2.11. As remarked in [18], since the generic member of the moduli space of $K3$ surfaces has Picard group of rank 1, this result covers almost all $K3$ surfaces.

3. Arakelov Geometry

3.1. Let K be a number field and let $\mathcal{O}_K$ be its ring of integers, and let S_∞ be the set of complex embeddings of K. If $\mathcal{M}$ is a torsion-free $\mathcal{O}_K$-module of finite rank such that, for all $\sigma \in S_\infty$, the corresponding complex vector space $M_\sigma = \mathcal{M} \otimes_{\mathcal{O}_K} \mathbb{C}$ is equipped with a norm $|\cdot|_\sigma$, we may think of $\mathcal{M}$ as a free $\mathbb{Z}$-module equipped with the norm $|\cdot|$on $M_\sigma = \mathcal{M} \otimes_{\mathcal{O}_K} \mathbb{C} = \oplus_{\sigma \in S_\infty} M_\sigma$ defined by $|\sum_{\sigma \in S_\infty} x_\sigma| = \sup_{\sigma \in S_\infty} |x_\sigma|_\sigma$ for $x_\sigma \in M_\sigma$, $\sigma \in S_\infty$. In particular, consider an hermitian vector bundle $\overline{E} = (\mathcal{E}, h)$ over $\mathrm{Spec}(\mathcal{O}_K)$ in the sense of [14]. In other words, $\mathcal{E}$ is a torsion-free $\mathcal{O}_K$-module of rank $N+1 < \infty$, and for all $\sigma \in S_\infty$, E_σ is equipped with a hermitian scalar product h, compatible with the isomorphism $E_\sigma \cong E_{\overline{\sigma}}$ induced by complex conjugation. We will then denote by $\|\cdot\|_\sigma$ the norm on E_σ and $\|\cdot\|$ the norm on $\mathcal{E} \otimes_{\mathbb{Z}} \mathbb{C}$ as above. If $N = 0$ then the *Arakelov degree* of $\overline{\mathcal{E}}$ is defined by

$$\widehat{\deg}\left(\overline{\mathcal{E}}\right) = \log(\#(\mathcal{E}/s \cdot \mathcal{O}_K)) - \sum_{\sigma \in S_\infty} \log \|s\|_\sigma,$$

where s is any non zero element of $\mathcal{E}$. In general, we define the *(normalized) Arakelov degree* of $\overline{\mathcal{E}}$ as $\widehat{\deg}_n(\overline{\mathcal{E}}) = \frac{1}{[K:\mathbb{Q}]}\widehat{\deg}\left(\det(\overline{\mathcal{E}})\right)$.
Let $\mathcal{E}^\vee$ be the dual $\mathcal{O}_K-$module of $\mathcal{E}$, and let $\mathbb{P}(\mathcal{E}^\vee)$ be the associated projective space representing lines in $\mathcal{E}^\vee$. Consider a closed subvariety $X \subseteq \mathbb{P}(E^\vee)$, where $E = \mathcal{E} \otimes K$, of dimension d, and let $\deg(X)$ be its (algebraic) degree with respect to the canonical line bundle $\mathcal{O}(1)$ on $\mathbb{P}(E^\vee)$. Let $h_{\mathcal{E}}(X) \in \mathbb{R}$ be the normalized Faltings height of the Zariski closure $\overline{X}$ of X in $\mathbb{P}(\mathcal{E})$, denoted by $h_F(X)/[K:\mathbb{Q}]$ in [3, $(3.1.1)$, $(3.1.5)$]. Let $\overline{\mathcal{O}(1)}$ be the canonical line bundle equipped with the metric induced by h, then

$$h_{\mathcal{E}}(X) = \frac{1}{[K:\mathbb{Q}]}\widehat{\deg}\left(\hat{c}_1(\overline{\mathcal{O}(1)})^{d+1}|\overline{X}\right) \in \mathbb{R}, \tag{3.1}$$

where $(.|.)$ is the bilinear pairing defined in loc. cit.

3.2. Let L be a finite field extension of K, and let $\overline{\mathcal{L}}_0, \cdots, \overline{\mathcal{L}}_N$ be hermitian line sub-bundles of $\overline{\mathcal{E}}$ such that $(\oplus_i \mathcal{L}_i)_L$ generates $\mathcal{E}_L$. There exist points $P_0 \cdots, P_N \in \mathbb{P}(\mathcal{E}_L^\vee)$ associated to the line bundles above such that $h(P_i) = -\widehat{\deg}(\overline{\mathcal{L}}_i)$ for $i = 0, \cdots, N$, [25, Theorem 5.2]. This implies that all $\widehat{\deg}(\overline{\mathcal{L}}_i)$ are non-positive. Assume that these line bundles are ordered by increasing Arakelov degree $\widehat{\deg}(\overline{\mathcal{L}}_0) \leqslant \cdots \leqslant \widehat{\deg}(\overline{\mathcal{L}}_N) \leqslant 0$. For $i = 0, \cdots, N$ define $s_i = -\widehat{\deg}(\overline{\mathcal{L}}_i) + \widehat{\deg}(\overline{\mathcal{L}}_N)$, and put $\mathbf{s} = (s_0, \cdots, s_N) \in \mathbb{R}^{N+1}$.
Let $x_0, \cdots, x_N$ be nonzero sections of the line bundles $\overline{\mathcal{L}_0}, \cdots, \overline{\mathcal{L}_N}$, respectively, that give an embedding $X \to \mathbb{P}^N$.
Further, let w be a weight function on $E = \mathcal{E}_K$ with weights $r_0 \geqslant \cdots \geqslant r_N = 0$, and $\mathbf{r} = (r_0, \cdots, r_N)$. We get the following variation of [22, Theorem 1]:

Theorem 3.3. *Assume there exists a continuous function $\psi : \mathbb{R}^{N+1} \to \mathbb{R}$ such that $\psi(tx) = t\psi(x)$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^{N+1}$, and such that*

$$e_w(X) \leqslant \psi(\mathbf{r}).$$

Then the following inequality holds:

$$h_{\mathcal{E}}(X) \geqslant (d+1)\deg(X)\widehat{\deg}(\overline{\mathcal{L}}_N) - \psi(\mathbf{s}).$$

Proof: This is a reinterpretation of the proof of of [26, Theorem 4.4] (see [26, 4.8]) using the language of of [22, Theorem 1]. □

Corollary 3.4. *Let X be a surface and $\mathcal{E}$ be a $\mathcal{O}_K$-module with $\mathcal{E}_K = H^0(X, \mathcal{O}(C))$, where C is a pseudo ample divisor. Then*

$$h_{\mathcal{E}}(X) \geqslant (d+1)\deg(X)\widehat{\deg}(\overline{\mathcal{L}}_N) - \sum_{k=0}^{m} \left(\widehat{\deg}(\overline{\mathcal{L}}_{j_{k+1}}) - \widehat{\deg}(\overline{\mathcal{L}}_{j_k})\right)\left(e_{j_k} + e_{j_k j_{k+1}} + e_{j_{k+1}}\right).$$

Proof. Straightforward consequence of Theorem 3.3 and (2.9). □

Corollary 3.5. *Assume that X is a $K3$ surface, and $\mathcal{E}$ be a $\mathcal{O}_K$-module with $\mathcal{E}_K = H^0(X, \mathcal{O}(C))$, where C is a pseudo ample divisor. For $i = 0, \cdots, N$, let V_i be the linear space of the filtration (2.8). Suppose that no curve is contained in the base locus of V_i, for all $i = 0, \cdots, N$. Then*

$$\begin{aligned} \frac{1}{2} h_{\mathcal{E}}(X) \quad \geqslant \quad & \max\Big\{(N-1)\Big(\widehat{\deg}(\overline{\mathcal{L}}_0) + 2\widehat{\deg}(\overline{\mathcal{L}}_N)\Big), \\ & -2\Big(\widehat{\deg}(\overline{\mathcal{L}}_0) + 2\widehat{\deg}(\overline{\mathcal{L}}_N)\Big) + 3\sum_{i=0}^{N} \widehat{\deg}(\overline{\mathcal{L}}_i)\Big\}. \end{aligned} \tag{3.2}$$

Proof: From the assumption that no curve is contained in the base locus of V_i, for all $i = 0, \cdots, N$, given in (2.8) we know that we can estimate the degree of contact with the

help of the estimate (2.9). By Theorem 3.3 and Riemann-Roch (2.10) we get

$$h_{\mathcal{E}}(X) \geqslant 6(N-1)\widehat{\deg}(\overline{\mathcal{L}}_N) - \min\left\{-4s_0 + 6\sum_{i=0}^{N} s_i, 2(N-1)s_0\right\},$$

where $s_i = \widehat{\deg}(\overline{\mathcal{L}}_N) - \widehat{\deg}(\overline{\mathcal{L}}_i)$. Expanding of this formula we get (3.2), which concludes the proof. □

Corollary 3.6. *Assume that the linear space defined by the vanishing of the last $d+1$ sections $x_{N-d}, \cdots, x_N$ does not meet X. Then*

$$h_{\mathcal{E}}(X) \geqslant \deg(X) \sum_{i=N-d}^{N} \widehat{\deg}(\overline{\mathcal{L}}_i)$$

Proof. Follows from the identity (2.7) and Theorem 3.3. □

3.7. We recall the definition of the normalized height $\hat{h}(X)$ form the introduction:

$$\hat{h}(X) = \inf_{\mathcal{E}\in\mathcal{V}} \left(\frac{h_{\mathcal{E}}(X)}{(d+1)\deg(X)} - \frac{\widehat{\deg}(\overline{\mathcal{E}})}{N+1}\right),$$

where $\mathcal{V}$ is the direct limit of the set $\mathcal{V}_K$ of $\mathcal{O}_K$-modules $\mathcal{E}$ with an identity $\mathcal{E}_K \cong K^{N+1}$ as K varies in the set of number fields. We will prove in this section that under some conditions on X the height $\hat{h}(X)$ is non-negative, as it is true in the function field case (see [5], [1]). The idea of the proof is the same of [22], the main difference relies in the direct use of the minima of X at the place of the minima of the projective space.

3.8. According to [26, (5.2)] (see also [6, Théorème 3.1]) we know that for all $\varepsilon > 0$ the set

$$X\left(\frac{h_{\mathcal{E}}(X)}{\deg(X)} + \varepsilon\right) = \left\{x \in X\left(\overline{\mathbb{Q}}\right) \,\middle|\, h(x) \leqslant \frac{h_{\mathcal{E}}(X)}{\deg(X)} + \varepsilon\right\} \tag{3.3}$$

is Zariski dense.

Proposition 3.9. *Assume that X is not contained in any proper linear subspace of $\mathbb{P}(E^\vee)$. Then for all $\varepsilon > 0$ there exist a finite field extension L of K and $N+1$ hermitian line subbundles $\overline{\mathcal{L}}_0, \cdots, \overline{\mathcal{L}}_N$ of $\overline{\mathcal{E}}$ such that*

(1) $\oplus_{i=0}^{N}\mathcal{L}_i$ *generates generically $\mathcal{E}$ overl L,*
(2) *The arithmetic degrees of this subbundles satisfy*

$$-\frac{1}{N+1}\sum_{i=0}^{N}\widehat{\deg}(\overline{\mathcal{L}}_i) \leqslant \frac{h_{\mathcal{E}}(X)}{\deg(X)} + \varepsilon. \tag{3.4}$$

Proof. Consider the $(N+1)$-fold Segre immersion φ of $X^{N+1} \subseteq \mathbb{P}(E^\vee)^{N+1}$ into the projective space $\mathbb{P}((E^\vee)^{\otimes(N+1)})$. From [3, $(2.3.19)$] we have

$$h_{(\mathcal{E})^{\otimes(N+1)}}\left(\varphi_*(X^{N+1})\right) = (N+1)\deg(X)^N h_{\mathcal{E}}(X),$$

and

$$\deg\left(\varphi_*(X^{N+1})\right) = \deg(X)^{N+1}.$$

Applying (3.3) to $\varphi_*(X^{N+1})$ we have that the set $\varphi_*(X^{N+1})\left(\frac{h_{\mathcal{E}}(\varphi_*(X))}{\deg(\varphi_*(X))} + \varepsilon\right)$ is Zariski dense. Since φ is an isomorphism, this set is in homeomorphic to

$$X^{N+1}\left((N+1)\frac{h_{\mathcal{E}}(X)}{\deg(X)} + \varepsilon\right),$$

which is therefore Zariski dense, too. Further, notice that the condition (1) is obviously open, since it corresponds to the non-vanishing of the maximal exterior power of the direct sum of the line subbundles. Hence (1) defines a Zariski open subset of

$$X^{N+1}\left((N+1)\frac{h_{\mathcal{E}}(X)}{\deg(X)} + \varepsilon\right),$$

which is itself Zariski dense. This implies that the set of points with (1) and (3) is Zariski dense, hence non-empty. Let $P = (P_0, \cdots, P_N)$ be a point there, and L be the field $K(P_0, \cdots, P_N)$, then $P \in \mathbb{P}(E^\vee)^{N+1}(L)$. Further, we let $\mathcal{L}_i$ the line subbundle of $\mathcal{E}$ associated to P_i, $i = 0, \cdots, N$. By construction they satisfy (1) and (2). This concludes the proof of the Proposition. □

Proof of Theorem 1.1. According to [26, Proposition 4.2] it suffices to prove that, for any number field K, $h_{\mathcal{E}}(X)/((d+1)\deg X) \geqslant 0$ for any hermitian $\mathcal{O}_K$-module $\overline{\mathcal{E}}$ such that $\det \mathcal{E} = 0$. By Proposition 3.9 for any $\varepsilon > 0$ there exist a finite field extension L of K and $N+1$ hermitian line subbundles $\overline{\mathcal{L}}_0, \cdots, \overline{\mathcal{L}}_N$ of $\overline{\mathcal{E}}$ such that $\oplus_{i=0}^N \mathcal{L}_i$ generates $\mathcal{E}$ over L and (3.4) holds. For each $i = 0, \cdots, N$ let x_i be a section of $\mathcal{L}_i$. These sections give an embedding $X \to \mathbb{P}^N$. Assume that the line bundles are ordered by increasing arithmetic degree

$$\widehat{\deg}(\overline{\mathcal{L}}_0) \leqslant \cdots \leqslant \widehat{\deg}(\overline{\mathcal{L}}_N) \leqslant 0.$$

Since X is semistable with respect to the sections $x_0, \cdots, x_N$ and to any $(N+1)$-tuple of integers with $r_0 \geqslant \cdots \geqslant r_N \geqslant 0$, we have

$$\frac{e_w(X)}{(d+1)\deg(X)} \leqslant \frac{1}{N+1}\sum_{i=0}^N r_i.$$

Hence, from Theorem 3.3 we get

$$\begin{aligned}\frac{h_{\mathcal{E}}(X)}{(d+1)\deg(X)} &\geqslant \widehat{\deg}(\overline{\mathcal{L}}_N)+\frac{1}{N+1}\sum_{i=0}^{N}\left(\widehat{\deg}(\overline{\mathcal{L}}_i)-\widehat{\deg}(\overline{\mathcal{L}}_N)\right)\\ &= \frac{1}{N+1}\sum_{i=0}^{N}\widehat{\deg}(\overline{\mathcal{L}}_i).\end{aligned}$$

Since $\overline{\mathcal{L}}_0, \cdots \overline{\mathcal{L}}_N$ satisfy (3.4) this yields

$$\frac{h_{\mathcal{E}}(X)}{(d+1)\deg(X)} \geqslant -\frac{h_{\mathcal{E}}(X)}{\deg(X)} - \varepsilon,$$

whence

$$\frac{h_{\mathcal{E}}(X)}{(d+1)\deg(X)} \geqslant -\frac{\varepsilon}{d+2}.$$

But $\varepsilon > 0$ can be chosen arbitrarily small, hence this concludes the proof of (1.2). □

Aknowledgements. It is a great pleasure to thank Jean-Benoît Bost for sharing with me his ideas on Arithmetic and Geometric Invariant Theory. Thanks are also due to Christophe Soulé for many useful comments and Jan-Hendrik Evertse for his constant encouragement and the explanation of the proof of **2.6**.

REFERENCES

[1] J.-B. BOST, Semi-stability and heights of cycles, *Invent. Math.*, **118** (1994) 223–252.

[2] J.-B. BOST, Intrinsic Heights of Stable Varieties and Abelian Varieties, *Duke Math. Journal*, **82** (1996) 21–70.

[3] J.-B. BOST, H. GILLET, C. SOULÉ, Heights of projective varieties and positive green forms, *Journ. of the AMS*, **7** (1994) 903–1027.

[4] J.-F. BURNOL, Remarques sur la stabilité en arithmétique, *Int. Math. Res. Notices*, **6** (1992) 117–127.

[5] M. CORMALBA, J. HARRIS, Divisor classes associated to families of stable varieties; with applications to the moduli space of curves, *Ann. Scint. Ec. Norm Sup.*, **21** (1988) 455–475.

[6] S. DAVID, P. PHILIPPON, Minorations des hauteurs normalisées des sous-variétés de variétés abeliennes, In *Number Theory, Tiruchirapalli* 1996, ed. V. K. Murty, M. Waldschmidt, *Contemp. Math.*, **210** (1997), 333–364.

[7] S. K. DONALDSON, Scalar Curvature and Projective Embeddings I, *Journ. Diff. Geom.*, **59** (2001) 479–522.

[8] J.-H. EVERTSE, R. G. FERRETTI, Diophantine Inequalities on Projective Varieties, *Int. Math. Res. Not.* , **2002:25** (2002) 1295–1330.

[9] R. G. FERRETTI, Mumford's Degree of Contact and Diophantine Approximations, *Comp. Math.*, **121** (2000) 247–262.

[10] R. G. FERRETTI, Diophantine Approximations and Toric Deformations, *Duke Math. J.*, **118** (2003) 493–522.

[11] W. FULTON, Intersection Theory, Erg. Math. Grenzgeb. **2**, Springer Verlag, 1984.

[12] C. GASBARRI, Heights and geometric invariant theory, *Forum. Math.*, **12** (2000) 135–153.

[13] D. GIESEKER, Global moduli for surfaces of general type, *Invent. Math.*, **43** (1977) 233–282.

[14] H. GILLET, C. SOULÉ, Characteristic classes for algebraic vector bundles with Hermitian metrics, *Annals of Math.*, **131** (1990) 163—203 and 205–238.

[15] W.V.D. HODGE, D. PEDOE, Methods of algebraic geometry, vol. II, Cambridge Univ. Press, Cambridge, 1952.

[16] M. M. KAPRANOV, B. STURMFELS, V. ZELEVINSKY, Chow polytopes and general resultants, *Duke Math. J.*, **67** (1992) 189–218.

[17] I. MORRISON, Projective Stability of Ruled Surfaces, *Invent. Math.*, **56** (1980) 269–304.

[18] I. MORRISON, Stability of Hilbert points of generic $K3$ surfaces, Preprint **401**, Centre de Recerca Matematica, Barcelona.

[19] D. MUMFORD, Stability of Projective Varieties, *Enseign. Math.*, **XXIII** (1977) 39–110.

[20] P. PHILIPPON, Sur des hauteurs alternatives III, *J. Math. Pures Appl.*, **74:4** (1995) 345–365.

[21] D. H. PHONG, J. STURM, Scalar Curvature, Moment Maps, and the Deligne Pairing Preprint math.DG/0209098.

[22] C. SOULÉ, Successive minima on arithmetic varieties, *Comp. Math.*, **96** (1995) 85–98.

[23] B. SAINT-DONAT, Projective models of $K3$ surfaces, *Amer. J. Math.*, **96** (1974) 602–639.

[24] E. VIEHWEG, Quasi-projective moduli for polarized manifolds, Erg. Math. Grenzgeb. **30**, Springer Verlag, 1995.

[25] S. ZHANG, Positive line bundles on arithmetic varieties, *Journ. Amer. Math. Soc.*, **8** (1995) 187–221.

[26] S. ZHANG, Heights and reductions of semi-stable varieties, *Comp. Math.*, **104** (1996) 77–105.

[27] S. ZHANG, Geometric reductivity at archimedean places, *Intern. Math. Res. Notices*, **10** (1994) 425–433.

UNIVERSITÁ DELLA SVIZZERA ITALIANA, VIA BUFFI 19, CH-6900 LUGANO, SWITZERLAND

DEPARTEMENT MATHEMATIK, ETH ZENTRUM, CH-8092 ZÜRICH, SWITZERLAND
E-mail address: `roberto.ferretti@lu.unisi.ch, ferretti@math.ethz.ch`